Complete MATH

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CLASS 11&12th

CLASS 11th

Complex Numbers

01. Why we Need Complex Numbers ?

The equations of the form $x^2+1=0$, $x^2+4=0$ etc. are not solvable in R i.e. there is no real number whose square is a negative real number. Euler was the first mathematician to introduce the symbol i (iota) for the square root of -1 i.e. a solution of $x^2+1=0$ with the property $i^2=-1$. He also called this symbol as the imaginary unit.

02. Integral Powers of Iota (i)

Positive integral power of i: We have,

$$i=\sqrt{-1}$$

$$\therefore \quad i^2=-1$$

$$i^3=i^2\times i=-i$$

$$i^4=\left(i^2\right)^2=(-1)^2=1$$

In order to compute i^n for $n>4$, we divide n by 4 and obtain the remainder r. Let m be the quotient when n is divided by 4. Then,

$$n=4m+r, \text{ where } 0\le r<4$$

$$\Rightarrow \quad i^n=i^{4m+r}=\left(i^4\right)^4 i^r=i^r$$

Thus, the value of i^n for $n>4$ is i^r, where r is the remainder when n is divided by 4.

Negative integral powers of i:

By the law of indices, we have,

$$i^{-1}=\frac{1}{i}=\frac{i^3}{i^4}=i^3=-i$$

$$i^{-2}=\frac{1}{i^2}=\frac{1}{-1}=-1$$

$$i^{-3}=\frac{1}{i^3}=\frac{i}{i^4}=i$$

$$i^{-4}=\frac{1}{i^4}=\frac{1}{1}=1$$

If $n>4$, them

$$i^{-n}=\frac{1}{i^n}=\frac{1}{i^{r\,\prime}} \text{ where } r \text{ is the remainder when } n \text{ is divided by 4.}$$

NOTE ☞ i^0 is defined as 1.

Properties of Iota

I. Periodic properties of I

$i^{4n}=1, i^{4n+1}=i,\ i^{4n+2}=-1, i^{4n+3}=-i\ \forall\ n\in Z$

II. Sum of four consecutive power terms of i is zero.

i.e., $i^n+i^{n+1}+i^{n+2}+i^{n+3}=0\ \forall n\in Z$

Example Evaluate :
i^{35}

Sol. 135 leaves remainder as 3 when it is divided by 4.
$\therefore \quad i^{35} = i^3 = -i$

03. Imaginary Quantities

The square root of a negative real number is called an imaginary quantity or an imaginary number.
For example, $\sqrt{-3}, \sqrt{-4}, \sqrt{-9/4}$ etc. are imaginary quantities.

RESULT
If a, b are positive real numbers, then $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$.

NOTE ☞ (1) For any two real numbers $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either positive or zero. In order words, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is not valid if a and b both are negative.
(2) For any positive real number a, we have $\sqrt{-a} = \sqrt{-1 \times a}\, \sqrt{-1} \times \sqrt{a} = i\sqrt{a}$.

04. Complex Numbers

COMPLEX NUMBER
If a, b are two real numbers, then a number of the form $a + ib$ is called a complex number.
For example, $7+2i, -1+i, 3-2i, 0+2i, 1+0i$ etc. are complex numbers.

Real and imaginary parts of a complex number: If $z = a + ib$ is a complex number, then $'a'$ is called the real part of z and $'b'$ is known as the imaginary part of z. The real part of z is denoted by Re (z) and the imaginary part by Im (z).
Example $z = 3-4i$, then Re $(z) = 3$ and Im $(z) = -4$.

Purely real and purely imaginary complex numbers: A complex number z is purely real if its imaginary part is zero i.e. Im $(z) = 0$ and purely imaginary if its real part is zero i.e. Re $(z) = 0$.

Set of complex numbers: The set of all complex numbers is denoted by C i.e.
$C = \{a + ib : a, b \in R\}$.

NOTE ☞ Since a real number $'a'$ can be written as $a + 0i$. Therefore, every real number is a complex number. Hence, $R \subset C$, where R is the set of all real numbers.

05. Equality of Complex Numbers

Definition Two complex numbers $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ and equal if $a_1 = a_2$ and $b_1 = b_2$ i.e. $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

Thus, $z_1 = z_2 \Leftrightarrow Re(z_1) = Re(z_2)$ and Im $(z_1) =$ Im (z_2).

06. Algebra of Complex Numbers

ADDITION

Definition Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ be two complex numbers. Then their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$. The sum $z_1 + z_2$ is a complex number such that Re $(z_1 + z_2)$ = Re (z_1) + Re (z_2) and Im $(z_1 + z_2)$ = Im (z_1) + Im (z_2)

PROPERTIES OF ADDITION OF COMPLEX NUMBERS

(i) Addition is Commutative: For any two complex numbers z_1 and z_2, we have $z_1 + z_2 = z_2 + z_1$.

(ii) Addition is Associative: For any three complex numbers z_1, z_2, z_3, we have $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) +$

(iii) Existence of Additive Identity: The complex number $0 = 0 + i0$ is the identity element for addition i.e. $z + 0 = z = 0 + z$ for all $z \in C$. The complex number $0 = 0 + i0$ is the identity element for addition.

(iv) Existence of Additive Inverse: For any complex number $z = a + ib$, there exists $-z = (-a) + i(-b)$ such that $z + (-z) = 0 = (-z) + z$. The complex number $-z$ is called the additive inverse of z.

07. Subtraction of Complex Numbers

Definition Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ be two complex numbers. Then the subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and $-z_2$.

Thus, $z_1 - z_2 = z_1 + (-z_2) = (a_1 + ib_1) + (-a_2 - i b_2) = (a_1 - a_2) + i(b_1 - b_2)$

08. Multiplication of Complex Numbers

Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ be two complex numbers. Then the subtraction of z_1 with z_2 is denoted by $z_1 z_2$ and is defined as the complex number $(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$.

Thus, $z_1 z_2 = (a_1 + i b_1)(a_2 + i b_2)$

$\Rightarrow \quad z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$\Rightarrow \quad z_1 z_2 = [\text{Re}(z_1)\ \text{Re}(z_2) - \text{Im}(z_1)\ \text{Im}(z_2)] + i\ [\text{Re}(z_1)\ \text{Im}(z_2) + \text{Re}(z_2)\ \text{Im}(z_1)]$