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Complete MATH

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CLASS 11&12th

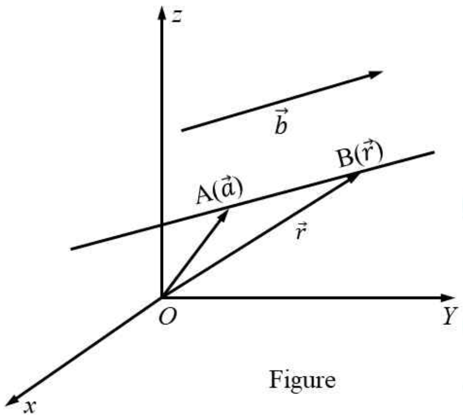
CLASS 12th

3-D Geometry

01. Straight Line In Space

Vector And Cartesian Equation Of A Line

Theorem I *The vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is scalar.*



Figure

Remark 1 *In the above equation $\vec{r}$ is the position vector of any point P (x, y, z) on the line. Therefore, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.*

Remark 2 *The position vector of any point on the line is taken as $\vec{a} + \lambda\vec{b}$.*

Theorem II *The cartesian equation of a straight line passing through a fixed point (x_1, y_1, z_1) and having direction ratios proportional to a, b, c is given by*

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Remark 1 *The parametric equations of the line* $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ *are* $x = x_1 + a\lambda,\ y = y_1 + b\lambda,\ z = z_1 + c\lambda$, *where λ is the parameter.*

Remark 2 *The coordinates of any point on the line* $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ *are* $(x_1 + a\lambda,\ y_1 + b\lambda,\ z_1 + c\lambda)\ \lambda \in R$.

Remark 3 *Since the direction cosines of a line are also direction ratios. Therefore, equation of a line passing through (x_1, y_1, z_1) and having direction cosines l, m, n is*

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Remark 4 *Since x, y and z-axes pass through the origin and have direction cosines* 1, 0, 0; 0, 1, 0 *and* 0, 0, 1 *respectively. Therefore, their equations are*

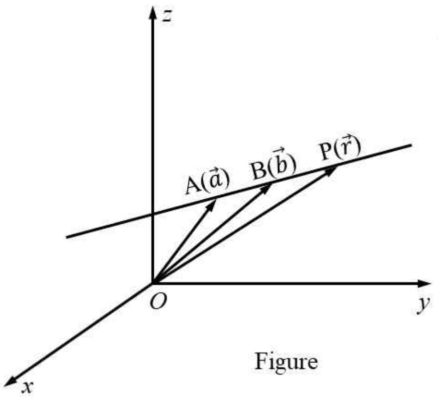
$$x\text{-axis}: \frac{x-0}{1}=\frac{y-0}{0}=\frac{z-0}{0} \quad \text{or, } y=0 \text{ and } z=0$$

$$y\text{-axis}: \frac{x-0}{0}=\frac{y-0}{1}=\frac{z-0}{0} \quad \text{or, } x=0 \text{ and } z=0$$

$$z\text{-axis}: \frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{1} \quad \text{or, } x=0 \text{ and } y=0$$

Theorem III *The vector equation of a line passing through two points with position vector* $\vec{a}$ *and* $\vec{b}$ *is*

$$\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$$



Figure

Theorem IV *The cartesian equations of a line passing through two given points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *are given by*

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Reduction of Cartesian Form of The Equation Of A Line to Vector Form And Vice-Versa

Cartesian to Vector Let the cartesian equation of a line be

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Then, the vector form of line (i) is

$$\vec{r}=\vec{a}+\lambda\vec{m}$$

or, $\vec{r}=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k})$, where λ is a parameter.

Vector To Cartesian Let the vector equation of a line be

$$\vec{r}=\vec{a}+\lambda\vec{m}$$

where, $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$, $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$ and λ is a parameter.

Then, $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}=\lambda$ is the cartesian equation of the line.

Angle Between Two Lines

Vector Form Let the vector equations of the two lines be

$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$.

$$\cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|}$$

Condition of perpendicularity: If the lines $\vec{b_1}$ and $\vec{b_2}$ are parallel.

$\therefore \quad \vec{b_1} = \lambda\vec{b_2}$ for some scalar λ.

Cartesian Form Let the cartesian equations of the two lines be

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \qquad \text{...(i)}$$

and, $$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \qquad \text{...(ii)}$$

Let θ be the angle between (i) and (ii).

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}$$

Condition of perpendicularity : If the lines are perpendicular, then

$\vec{m_1}\cdot\vec{m_2} = 0 \Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

Condition of parallelism : If the lines are parallel, then $\vec{m_1}$ and $\vec{m_2}$ are parallel.

$\therefore \quad \vec{m_1} = \lambda\vec{m_2}$ for some scalar λ

$\Rightarrow \quad \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Intersection of Two Lines

To determine whether two lines intersect or not and in case they intersect the following algorithm is used to find their point of intersection.

Algorithm Let the two lines be

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \qquad \text{...(i)}$$

and, $$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \qquad \text{...(ii)}$$

Step I *Write the coordinates of general points on (i) and (ii). The coordinates of general points on (i) and (ii) are given*

$\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b-1} = \dfrac{z-z_1}{c_1} = \lambda$ and $\dfrac{x-x_2}{a_2} = \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{c_2} = \mu$ *respectively,*

i.e. $(a_1\lambda + x_1, b_1\lambda + y_1, c_1\lambda + z_1)$ *and* $(a_2\mu + x_2, b_2\mu + y_2, c_2\mu + z_2)$

Step II *If the line (i) and (ii) intersect, then they have a common point.*

$\therefore\ a_1\lambda + x_1 = a_2\mu + x_2,\ b_1\lambda + y_1 = b_2\mu + y_2$ *and* $c_1\lambda + z_1 = c_2\mu + z_2$

Step III *Solve any two of the equation of* λ *and* μ *obtained in step II. If the values of* λ *and* μ *satisfy the third equation, then the lines (i) and (ii) intersect. Otherwise they do not intersect.*