MATHEMATICS

CLASS NOTES FOR CBSE

Chapter 09. Sequences and Series

01. Sequence

A sequence is a function whose domain is the set N of natural numbers.
Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. The images of 1,2,3, ... n, ... under a sequence 'a' are generally denoted by a_1, a_2, a_3,... a_n,... respectively. a_1, a_2, a_3,... a_n,... are known as first term, second term ..., nth term,... respectively of the sequence. If an is the nth term of a sequence, 'a' then we write $a = < a_n >$.

Real Sequence *A sequence whose range is a subset of R is called a real sequence. In other words, a real sequence is a function with domain n and the range a subset of the set R of real numbers.*

For example, 1, 3, 5, ... is a sequence whose nth term is $(2n - 1)$.
∴ The sequence 1, 3, 5, 7, ... can be written as $a_n = 2n - 1$.

02. Arithmetic Progression

Definition *A sequence a_1, a_2, a_3, ..., a_n, ... is called an arithmetic progression (A. P.), if the difference of any term and the previous term is always same.*

i.e., $a_{n+1} - a_n$ = Constant (= d) for all $n \in N$
or, $a_{n+1} - a_n$ is independent of n.
The constant difference 'd' is called the common difference.
Example I 1, 4, 7, 10, ... is an *A.P.* whose first term is 1 and the common difference is equal to $4 - 1 = 3$.
Example II 11, 7, 3, – 1, ... is an A.P. whose first term is 11 and the common difference is equal to $7 - 11 = -4$.
To determine whether a sequence is an A.P. or not when its nth term is given, we may use the following algorithm:

Algorithm
Step I *Obtain* a_n
Step II Replace n by n + 1 in an to get an + 1
Step III Calculate an + 1 – an

Step IV *If* $a_{n+1} - a_n$ *is independent of n, the given sequence is an A.P. Otherwise it is not an A.P.*

General Term of An A.P.

Theorem *Let a be the first term and d be the common difference of an A.P. Then its nth term is* $a + (n - 1)d$ *i.e.* $a_n = a + (n - 1)d$.

Proof Let $a_1, a_2, a_3, a_4, ..., a_n, ...$ be the given A.P. Then,

$$a_1 = \text{First term} = a \Rightarrow a_1 = a + (1 - 1)\, d.$$

By the definition, we have

$$a_2 - a_1 = d \Rightarrow a_2 = a_1 + d \Rightarrow a_2 = a + d \Rightarrow a_2 = a + (2 - 1)\, d;$$

$$a_3 - a_2 = d \Rightarrow a_3 = a_2 + d \Rightarrow a_3 = (a + d) + d \Rightarrow a_3 = a + 2d$$

$$\Rightarrow a_3 = a + (3 - 1)\, d;$$

$$a_4 - a_3 = d \Rightarrow a_4 = a_3 + d \Rightarrow a_4 = (a + 2d) + d \Rightarrow a_4 = a + 3d$$

$$\Rightarrow a_4 = a + (4 - 1)\, d$$

Similarly, $a_5 = a + (5 - 1)d$, $a_6 = a + (6 - 1)d$, ..., $a_n = a + (n - 1)d$.

Hence, nth term of an A.P. with first term a and common difference d is $a_n = a + (n - 1)d$.

***n*th Term of An A.P. From The End**

Let a be the first term and d be the common difference of an A.P. having m terms. Then nth term from the end is (m – n +1)th term from the beginning.

$\therefore$ nth term from the end $= a_{m-n+1}$

$$= a + (m - n + 1 - 1)\, d = a + (m - n)d$$

Example *Which term of the sequence* 72, 70, 68, 66, ... *is* 40?

Solution Clearly, the given sequence is an A.P. with first term = 7 and common difference = – 2.

Let its nth term be 40. Then,

$$72 + (n - 1)(-2) = 40$$

$$\Rightarrow 72 - 2n + 2 = 40 \qquad [\because a_n = a + (n - 1)d]$$

$$\Rightarrow 2n = 34$$

$$\Rightarrow n = 17$$

Hence, 17^{th} term of the given sequence is 40.

(i) Selection of terms of An A.P.

Sometimes we require certain number of terms in A.P. The following ways of selecting terms are generally very convenient :

Number of terms	*Terms*	*Common Difference*
3	a – d, a, a + d	d
4	a – 3d, a – d, a + d, a + 3d	2d
5	a – 2d, a – d, a, a + d, a + 2d	d
6	a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d	2d

It should be noted that in case of an odd number of terms, the middle term is a and the common difference is d while in case of an even number of terms the middle terms are a – d, a + d and the common difference is $2d$.

(ii) Sum to *n* terms of An A.P.

The sum S_n of n terms of an A.P. with first term 'a' and common difference 'd' is given by

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

NOTE 1 ☞ In the formula Sn = $\frac{n}{2}[2a + (n-1)d]$, there are four quantities viz. S_n, a, n and d. If any three of these are known, the four can be determined. sometimes two of these quantities are given, in such cases remaining two quantities are provided by some other relation.

NOTE 2 ☞ *If the sum S_n of n terms of a sequence is given, then nth term a_n of the sequence can be determined by the following formula.*

Example *Find the sum of first 20 terms of an A.P., in which* 3^{rd} *term is 7 and* 7^{th} *term is two more than thrice of its* 3^{rd} *term.*

Solution Let a be the first term and d be the common difference of the given A.P. Then,

$a_3 = 7$ and $a_7 = 3a_3 + 2$ (given)

$\Rightarrow$ $a + 2d = 7$ and $a + 6d = 3(a + 2d) + 2$

$\Rightarrow$ $a + 2d = 7$ and $a = -1 \Rightarrow a = -1,\ d = 4$

Now, $S_{20} = \frac{20}{2}[2 \times -1 + (20-1) \times 4]$ [Using $S_n = \frac{n}{2}[2a + (n-1)d]$]

$\Rightarrow$ $S_{20} = \frac{20}{2}[-2 + 76] = 740$

(iii) Properties of Terms of an A.P.

Property 1 *If an constant if added to or subtracted from each term of an A.P. then the resulting sequence is also an A.P. with the same common difference.*

Proof Let $a_1, a_2, a_3, ...$be an A.P. with common difference d, and let k be a fixed constant which is added to each term of this A.P. Then, the resulting sequence is

$a_1 + k,\ a_2 + k,\ a_3 + k_1...$

Let $b_n = a_n + k$, $n = 1, 2,...$ Then, the new sequence is $b_1, b_2, b_3,...$

We have, $b_{n+1} - b_n = (a_{n+1} + k) - (a_n + k)$

$= a_{n+1} - a_n = d$ for all $n \in N$ [$\because$ $\langle a_n \rangle$ is a sequence with common difference d]

Thus, the new sequence is also an A.P. with common difference d.

Property 2 *If each term of an A.P. is multiplied or divided by a non-zero constant k, then the resulting sequence is also an A.P. with common difference kd or d/k, where d is the common difference of the given A.P.*

Proof Let $a_1, a_2, a_3, ...$be an A.P. with common difference d, and let k be a non-zero constant. Let $b_1, b_2, b_3,...$ be sequence obtained by multiplying each term of the given A.P. by k. Then,