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Determinants

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01. Determinants

DEFINITION

Every square matrix can be associated to an expression or a number which is known as its determinant. If $A=[a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det A or, $|A|$ or,

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & & & \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

DETERMINANT OF A SQUARE MATRIX OF ORDER 1

If $A=[a_{11}]$ is a square matrix of order 1, then the determinant of A is defined as

$$|A| = a_{11} \qquad \text{or,} \qquad |a_{11}| = a_{11}$$

DETERMINANT OF A SQUARE MATRIX OF ORDER 2

The determinant of a square matrix of order 2 is equal to the product of the diagonal elements minus the product of off-diagonal elements.

DETERMINANT OF A SQUARE MATRIX OF ORDER 3

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, then the expression

$$a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{32}\,a_{21} - a_{11}\,a_{23}\,a_{32} - a_{22}\,a_{13}\,a_{31} - a_{12}\,a_{21}\,a_{33}$$

is defined as the determinant of A i.e.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{32}\,a_{21} - a_{11}\,a_{23}\,a_{32} - a_{22}\,a_{31}\,a_{13} - a_{33}\,a_{12}\,a_{21} \quad \text{...(ii)}$$

or, $$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Rightarrow \quad |A| = a_{11}\left(a_{22}\,a_{33} - a_{23}\,a_{32}\right) - a_{12}\left(a_{33}\,a_{21} - a_{23}\,a_{31}\right) + a_{13}\left(a_{32}\,a_{21} - a_{22}\,a_{31}\right)$$

$$\Rightarrow \quad |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \qquad \text{[Using notation given in (i)]}$$

$$\Rightarrow \quad |A| = (-1)^{1+1}\,a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}\,a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}\,a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Thus the determinant of a square matrix of order 3 is the sum of the product of elements a_{1j} in first row with $(-1)^{1+j}$ times the determinant of a 2×2 sub-matrix obtained by leaving the first row and column passing through the element.

NOTE ☞ 1. Only square matrices have determinants. The matrices which are not square do not have determinants.
2. The determinant of a square matrix of order 3 can be expanded along any row or column.
3. If a row or a column of a determinant consists of all zeros, then the value of the determinant is zero.

DETERMINANT OF A SQUARE MATRIX OF ORDER 4 OR MORE

To evaluate the determinant of a square matrix of order 4 or more we follow the same procedure as discussed in evaluating the determinant of a square matrix of order 3.

02. Minors and Cofactors

Minor

Let $A=[a_{ij}]$ be a square matrix of order n. Then the minor M_{ij} of a_{ij} in A is the determinant of the square sub-matrix of order $(n-1)$ obtained by leaving i^{th} row and j^{th} column of A.

Cofactor

Let $A=[a_{ij}]$ be a square matrix of order n. Then, the cofactor C_{ij} of a_{ij} in A is equal to $(-1)^{i+j}$ times the determinant of the sub-matrix of order $(n-1)$ obtained by leaving i^{th} row and j^{th} column of A.

It follows from this definition that

$$C_{ij} = \text{Cofactor of } a_{ij} \text{ in } A=(-1)^{i+j}M_{ij}, \text{ where } M_{ij} \text{ is minor of } a_{ij} \text{ in } A.$$

Thus, we have

$$C_{ij}=\begin{cases} M_{ij} & \text{if } i+j \text{ is even} \\ -\mathrm{M_{ij}} & \text{if } i+j \text{ is odd}\end{cases}$$

EXPANSION OF DETERMINANT USING MINORS/CO-FACTORS

The value of determinant is defined as the sum of the product of elements of any row (column) by their corresponding co-factors.

$$\text{Let } \triangle=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix}$$

$$=\underbrace{a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}}_{\text{expanding along } R_1}$$

$$=a_{11}a_{22}a_{33}-a_{11}a_{32}a_{23}+a_{12}a_{31}a_{23}-a_{12}a_{21}a_{33}+a_{13}a_{21}a_{32}-a_{13}a_{31}a_{22}$$

Therefore the value of the determinant can be obtained

as $\triangle=a_{11}\,C_{11}+a_{12}\,C_{12}+a_{13}\,C_{13}$ (expanding along R_1)

or $\triangle=a_{12}\,C_{12}+a_{22}\,C_{22}+a_{32}\,C_{33}$ (expanding along C_2)

or $\triangle = a_{31}\,C_{31} + a_{32}\,C_{32} + a_{33}\,C_{33}$ (expanding along R_3)

In general, expanding along i^{th} row we get

$$\triangle = a_{i1}\,C_{i1} + a_{i2}\,C_{i2} + ... + a_{in}\,C_{in} = \sum_{k=1}^{n} a_{ik}\,C_{ik} \qquad (\text{for all } i = 1, 2, ...n)$$

And expanding along j^{th} column, we get

$$\text{or } \triangle = a_{1j}\,C_{1j} + a_{2j}\,C_{j2} + ... + a_{nj}\,C_{nj} = \sum_{k=1}^{n} a_{kj}\,C_{kj} \qquad (\text{for all } j = 1, 2, ...\text{or } n)$$

NOTE ☞

1. The expansion generates same value irrespective of its performance through any row or column.
2. The expansion contains 3! i.e., 6 terms which is the number of permutations of 1, 2, 3 in a line.

$$\text{i.e., } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \underbrace{(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3)}_{positive\ zone} - \underbrace{(a_1b_3c_2 + b_1a_2c_3 + c_1a_3b_2)}_{\neg ative\ zone}.$$

3. Each term is product of three entries of the determinant.
4. 3 terms are positive, 3 other are negative (even and odd permutations).
5. Each entry of the determinant $\triangle$ once appears in the positive zone and once in the negative zone. For instance a_1 appears as in the first and fourth term.
6. A square matrix is a singular matrix if its determinant is zero. Otherwise, it is a non-singular matrix.

03. Properties of Determinants

Property 1

Let $A = [a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with their cofactors in always equal to $|A|$ or, det (A) i.e.

$$\sum_{j=1}^{n} a_{ij}\,C_{ij} = |\,A\,| \text{ and } \sum_{i=1}^{n} a_{ij}\,C_{ij} = |\,A\,|.$$

Property 2

Let $A = [a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with the cofactors of the corresponding elements of some other row (column) is zero i.e.

$$\sum_{j=1}^{n} a_{ij}\,C_{ij} = 0 \text{ and } \sum_{i=1}^{n} a_{ij}\,C_{ik} = 0.$$

Property 3

Let $A = [a_{ij}]$ be a square matrix of order n, then $|\,A\,| = |\,A^T|$.

or, the value of a determinant remains unchanged if its rows and columns are interchanged.