MATHEMATICS

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01. Normals

Theorem 1 (Point Form) *The equation of the normal to the parabola* $y^2 = 4ax$ *at a point* (x_1, y_1) *is given by*

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

Proof We know that $yy_1 = 2a(x + x_1)$ is the equation of the tangent to $y^2 = 4ax$ to point (x_1, y_1). The normal at any points is a line perpendicular to the tangent and passing thorugh the point of contact. So, the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

OR The equation of the normal at (x_1, y_1) is given by

$$y - y_1 = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}(x = x_1) \quad \text{...(i)}$$

Here, $y^2 = 4ax$

$\therefore \quad 2y\frac{dy}{dx} = 4a$ [on differentiating *w.r.* to x]

$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1 y_1)} = \left(\frac{2a}{y1}\right)$

Substituting the value of $\frac{dy}{dx}$ in (i), we obtain

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

as the equation of the normal to $y^2 = 4ax$ at (x_1, y_1).

Remark 1 *The equation of the normals to all standards form of parabola at* (x_1, y_1) *are given below for ready reference:*

Equation of the parabola	*Equation of the normal*
$y^2 = 4ax$	$y - y_1 = -\frac{y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$x - x_1 = -\frac{x_1}{2a}(y - y_1)$
$x^2 = -4ay$	$x - x_1 = \frac{x_1}{2a}(y - y_1)$

Theorem 2 (Parametric Form) *Prove that the equation of the normal to the parabola* $y^2 = 4ax$ *at point* $(at^2, 2at)$ *is given by*

$$y + tx = 2at + at^3$$

Proof We know that the equation of the normal to $y^2 = 4ax$ at (x_1, y_1) is given by

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

Here, $x_1 = at^2$ and $y_1 = 2at$.

So, the equation of the normal at $(at^2, 2at)$ is given by

$$y - 2at = -t(x - at^2)$$

or, $y + tx = 2at + at^3$

Remark 2 *The equation of normals to all standard forms of parabola in terms of parameter 't' are listed below for ready reference:*

Equation of parabola	*Parametric coordinates*	*Equation of normal*
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

Theorem 3 (Slope Form) *Prove that the equation of the normal to the parabola $y^2 = 4ax$ in terms of its slope m is given by*

$$y = mx - 2am - am^3$$

at the point $(am^2, -2am)$.

Proof We know that the equation of the normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is given

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

Let m be the slope of the normal. Then,

$$-\frac{y_1}{2a} = m \Rightarrow y_1 = -2am$$

Since (x_1, y_1) lies on $y^2 = 4ax$.

$\therefore\ y_1^2 = 4ax_1 \Rightarrow 4a^2m^2 = 4ax_1 \Rightarrow x_1 = am^2$.

Substituting the values of x_1 and y_1 in (i), we obtain

$$y + 2am = m(x - am^2)$$

or, $$y = mx - 2am - am^3$$

as the equation of the normal at $(am^2, 2am)$.

Remark 3 *The equation of normals to all standard forms of parabola in terms of parameter 't' are listed below for ready reference:*

Equation of the parabola	*Equation of the normal*	*Slope of the normal*	*Point of contact (Feet of the normal)*
$y^2 = 4ax$	$y = mx - 2am - am^3$	m	$(am^2, -2am)$
$y^2 = -4ax$	$y = mx + 2am + am^3$	m	$(-am^2, 2am)$
$x^2 = 4ay$	$x = my - 2am - am^3$	$1/m$	$(-2am, am^2)$
$x^2 = -4ay$	$x = my + 2am + am^3$	$1/m$	$(2am, -am^2)$

Remark 4 *It follows from the above discussion that the line* $y = mx + c$ *will be a normal to* $y^2 = 4ax$, *if* $c = -2am - am^3$.

The conditions of normality of different lines to various standard forms of parabola are listed below for ready reference:

Parabola	*Equation of the line*	*Slope of the line*	*Condition of normality*
$y^2 = 4ax$	$y = mx + c$	m	$c = -2am - am^3$
$y^2 = -4ax$	$y = mx + c$	m	$c = 2am + am^3$
$x^2 = 4ay$	$x = my + c$	$1/m$	$c = -2am - am^3$
$x^2 = -4ay$	$x = my + c$	$1/m$	$c = 2am + am^3$

Remark 5 *The equations of normals to the parabolas reducible to one of the standard forms are given below for ready reference:*

Parabola	*Equation of the normal*
$(y - k)^2 = 4a(x - h)$	$y - k = m(x - h) - 2am - am^3$
$(y - k)^2 = -4a(x - h)$	$y - k = m(x - h) + 2am + am^3$
$(x - h)^2 = 4a(y - k)$	$x - h = m(y - k) - 2am - am^3$
$(x - h)^2 = -4a(y - k)$	$x - h = m(y - k) + 2am + am^3$