SAMPLE PAPER

2019 JEE ADVANCED

MATHEMATICS

SET–2

Roll No. | | | | | | | |

ANSWER AND SOLUTION

1. (a, b, c, d)

$z_1 = \cos\theta + i\sin\theta$

$z_2 = 2(\cos\phi + i\sin\phi)$

$\Rightarrow \quad |z_1 - 2z_2|^2 = (\cos\theta - 4\cos\phi)^2 + (\sin\theta - 4\sin\phi)^2$

$= 1 + 16 - 8\cos(\theta - \phi)$

$\Rightarrow \quad 9 \le |z_1 - 2z_2|^2 \le 25$

Similarly, $1 \le |z_1 + z_2| \le 3$, $|z_1 - z_2| \ge 1$ and $|z_1 - 3z_2| \ge 5$.

2. (a, b, c, d)

Put $(2x - 1)^{1/3} = y$ or $y^3 = (2x - 1)$.

The original equation becomes

$x^3 = 2y - 1$

$\therefore \quad x^3 - y^3 = -2(x - y)$

$\Rightarrow \quad x - y = 0$ or $x^2 + y^2 + xy = -2$

$\Rightarrow \quad x = y$ as other equation is not possible

$\therefore \quad 2x - 1 = x^3$ or $x^3 - 2x + 1 = 0$

$\Rightarrow \quad x = 1, -(1 + \sqrt{5})/2, -(1 - \sqrt{5})/2$

3. (a, b)

Let roots be $\alpha - \beta$, α, $\alpha + \beta$, so that

$$3\alpha = -b \Rightarrow \alpha = -b/3$$

$$\therefore \; -\frac{b^3}{27} + \frac{b^3}{9} - \frac{bc}{3} + d = 0$$

$\Rightarrow 2b^3 - 9bc + 27d = 0.$

Next, roots be α/β, α, $\alpha\beta$, so that

$\alpha^3 = -d$ or $\alpha = (-d)^{1/3}$

$\therefore -d + b(-d)^{2/3} + c(-d)^{1/3} + d = 0$

$\Rightarrow b^3 d = c^3.$

4. (a, b, c)

$- - - - - -$ (even)
9 10 10 10 5 (ways)
So, $x = 9 \times 10^3 \times 5 = 45000$ ways to fill

$- - - - - -$ (odd)
9 10 10 10 5 (ways)
So, $y = 9 \times 10^3 \times 5 = 45000$ ways to fill

5. (b, c)

$$\frac{{}^{n}C_r}{{}^{r+2}C_r} = \frac{n!}{r!\,(n-r)!}\;\frac{r!\,2!}{(r+2)!} = \frac{2(n!)}{(r+2)!\,(n-r)!}$$

$$= \frac{2}{(n+1)(n+2)}\;\frac{(n+2)!}{(r+2)!\,[(n+2)-(r+2)]!}$$

$$= \frac{2}{(n+1)(n+2)}\;{}^{n+2}C_{r+2}.$$

Thus,

$$S = \sum_{r=0}^{n} (-2)^r \left(\frac{{}^{n}C_r}{{}^{r+2}C_r}\right)$$

$$= \frac{2}{(n+1)(n+2)} \sum_{r=0}^{n} (-2)^r \,.\, {}^{n+2}C_{r+2}$$

$$= \frac{2}{(n+1)(n+2)} \sum_{s=2}^{n+2} (-2)^{s-2} \,.\, {}^{n+2}C_s \text{ [put } r + 2 = s]$$

$$= \frac{2}{4(n+1)(n+2)} \sum_{s=2}^{n+2} {}^{n+2}C_s\,(-2)^s$$

$$= \frac{1}{2(n+1)(n+2)} \times \left[\sum_{s=0}^{n+2} {}^{n+2}C_s\,(-2)^s - {}^{n+2}C_0\,(-2)^0 - {}^{n+2}C_1\,(-2)^1\right]$$

$$= \frac{1}{2(n+1)(n+2)} \left[(1-2)^{n+2} - 1 + 2(n+2)\right]$$

$$= \frac{1}{2(n+1)(n+2)} \left[2n + 3 + (-1)^n\right]$$

But $2n + 3 + (-1)^n = \begin{cases} 2(n+2) \text{ if } n \text{ is even,} \\ 2(n+1) \text{ if } n \text{ is odd} \end{cases}$

Thus, $S = \begin{cases} \dfrac{1}{n+1} \text{ if } n \text{ is even,} \\ \dfrac{1}{n+2} \text{ if } n \text{ is odd} \end{cases}$

6. (a, c, d)

(a) $A * B = \frac{1}{2}(AB' + BA')$

$B * A = \frac{1}{2}(BA' + AB')$

So, $A * B = B * A$

(b) $A * A = \frac{1}{2}(AA' + AA') = AA' \neq A^2$

(c) $A * (B + C) = \frac{1}{2}(A(B + C)' + (B + C)A')$

$A * B + A * C = \frac{1}{2}(AB' + BA') + \frac{1}{2}(AC' + CA')$

$= \frac{1}{2}(A(B+C)' + (B+C)A')$

So, true

(d) $2A * I = (AI' + IA') = A + A'$

So, true

7. (b)

By applying the operation $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we get

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1 + 4\sin 6\theta \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1 + 4\sin 6\theta + \sin^2\theta \end{vmatrix} = 0$$

[use $C_3 \to C_3 + C_1$]

$\Rightarrow 1 + 4 \sin 6\theta + \sin^2\theta + \cos^2\theta = 0$

$\Rightarrow 4 \sin 6\theta = -2 \Rightarrow \sin 6\theta = \frac{-1}{2}$

8. (c)

P(4^{th} time in 7^{th} draw is white)

= P (1st 6 draws give 3 times white ball). P (7^{th} draw is white ball)

$= \quad {}^6C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)$

$= \quad 20 \times \frac{1}{64} \times \frac{1}{2} = \frac{10}{64} = \frac{5}{32}$

9. (c)

$f(\theta) = \sin\theta\ (\sin\theta + \sin 3\theta)$

$= \sin\ .\ 2\sin(2\theta)\cos(\theta)$

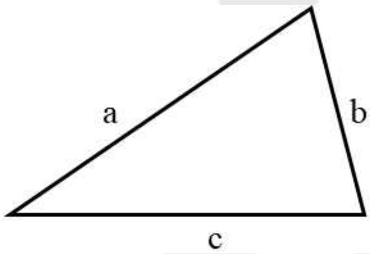
$= 4\sin^2\theta\cos^2\theta \geq 0$ for all real θ.

10. (b)

$a + b = x,\ ab = y$

$x^2 - c^2 = y$

$\frac{r}{R} = \frac{\Delta/s}{abc/4\Delta} = \frac{4\Delta^2}{abcs}$



$= \frac{a^2b^2\sin^2 c}{abc\ s}\left(\Delta = \frac{1}{2}ab\sin c\right)$

$= \frac{ab\sin^2 c}{sc} = \frac{2y\sin^2 c}{(x+c)c}$

$(\because\ 2s = x + c,\ y = ab)$

So, $\frac{r}{R} = \frac{2y}{(x+c)c}\sin^2 C$

when $\sin^2 C = 1 - \cos^2 C$

$= 1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2$

$= 1 - \left(\frac{(a+b)^2-2ab-c^2}{2ab}\right)^2$

$= 1 - \left(\frac{x^2-2y+y-x^2}{2y}\right)^2$

$= \frac{3}{4}$

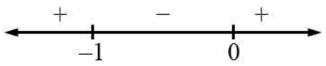
So,

$\frac{r}{R} = \frac{3y}{2c(x+c)}$

11. (c)

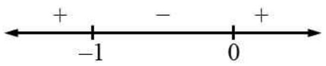
$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

$x(x + 1) \geq 0$



$x \leq -1$ or $x \geq 0$

$\underbrace{x^2+x+1 \geq 0}_{always\ \ true}$; $\underbrace{x^2+x+1 \leq 1}_{\substack{x^2+x \leq 0 \\ \Rightarrow x(x+1) \leq 0}}$



$-1 \leq x \leq 0$

Common domain : $x = 0, -1$

for $x = 0$, L.H.S. $= \tan^{-1} 0 + \sin^{-1} 1$

$= 0 + \frac{\pi}{2}$

$= \frac{\pi}{2}$

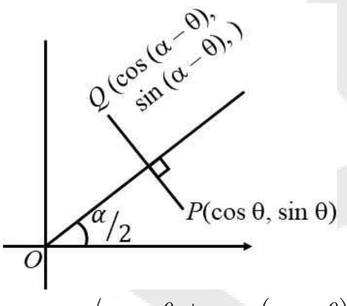
$=$ R.H.S.

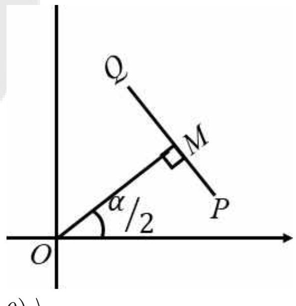
For $x = -1$, L.H.S. $= \tan^{-1} 0 + \sin^{-1} 1$

$= \frac{\pi}{2} =$ R.H.S.

$x = -1, 0$

12. (d)





$$M \equiv \left(\frac{\cos\theta + \cos(\alpha-\theta)}{2}, \frac{\sin\theta + \sin(\alpha-\theta)}{2}\right)$$

$$= \left(\cos\left(\frac{\alpha}{2}\right)\cos\left(\theta - \frac{\alpha}{2}\right), \sin\left(\frac{\alpha}{2}\right)\cos\left(\theta - \frac{\alpha}{2}\right)\right)$$

$\text{Slope}_{OM} = \tan \alpha/2$

$$\text{Slope}_{PQ} = \frac{\sin\theta - \sin(\alpha-\theta)}{\cos\theta - \cos(\alpha-\theta)}$$

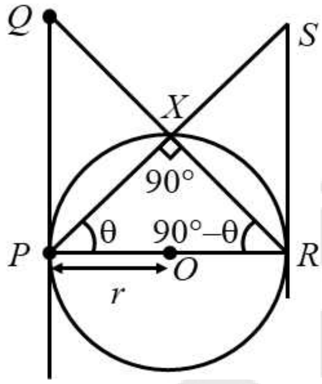
$$= \frac{2\sin\left(\theta - \frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{-2\sin\left(\frac{\alpha}{2}\right)\sin\left(\theta - \frac{\alpha}{2}\right)}$$

$$= -\cot\left(\frac{\alpha}{2}\right)$$

$Slope_{OM} \times Slope_{PQ} = -1$

PQ $\perp^r$ OM.

13. (a)



$\angle$PXR = 90° (angle in semi – circle)

Δ QPR, $\tan(90° - \theta) = \frac{PQ}{PR} = \frac{PQ}{2r}$

$\Rightarrow \cot\theta = \frac{PQ}{2r}$ ...(i)

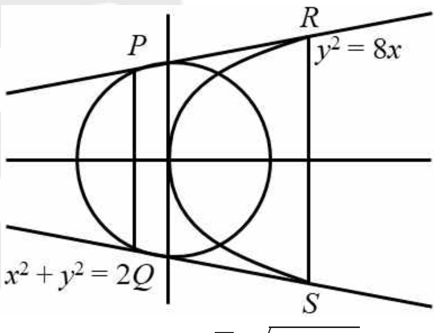
ΔPSR, $\tan\theta = \frac{RS}{PR} = \frac{RS}{2r}$ ...(ii)

$\Rightarrow$ (i) × (ii) $\Rightarrow 1 \; \frac{PQ.RS}{(2r)^2} = \frac{PQ.RS}{4r^2}$

$\Rightarrow 4r^2 = PQ \,.\, RS$

$\therefore 2r = \sqrt{PQ.RS}$

14. (d)



Let common tangent be $y = mx \pm \sqrt{2}\sqrt{1+m^2}$ is tangent to circle.

If it is tangent to parabola, $y = mx + \frac{2}{m}$

So,

$$\pm\sqrt{2}\times\sqrt{1+m^2}=\frac{2}{m}$$

$\Rightarrow$ $m^2(2+2m^2)=4$

$\Rightarrow$ $m^4+m^2-2=0$

$\Rightarrow$ $m^2=\dfrac{-1\pm\sqrt{1+8}}{2}=1,\,-2$

So, $m^2=1$ (acceptable)

$\Rightarrow$ $m=\pm 1$

So, tangent is : $y=\pm x\pm 2$

$\Rightarrow$ $y=x+2$ and $y=-x-2$

for R, $y^2=8x \Rightarrow (x+2)^2=8x$

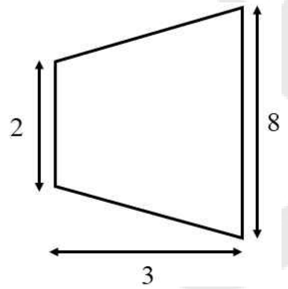
$\Rightarrow$ $x^2-4x+4=0$ $\Rightarrow (x-2)^2=0 \Rightarrow x=2$ and $y=\pm 4$

for P, $x^2+y^2=2 \Rightarrow x^2+(x+2)^2=2$

$\Rightarrow$ $2x^2+4x+2=0$ $\Rightarrow x^2+2x+1=0$

$\Rightarrow$ $(x+1)^2 \Rightarrow x=-1$, so $y=\pm 1$

$\therefore$ $R\equiv(2,4)$, $S\equiv(2,-4)$, $P\equiv(-1,1)$, $Q\equiv(-1,-1)$



Area $=\dfrac{1}{2}(2+8)\times 3$

$=15$

15. (a)

Normal to the required plane is perpendicular to the given 2 planes' normals

So, $\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 7 & 1 & 2\\ 3 & 5 & -6\end{vmatrix}=-16\hat{i}+48\hat{j}+32\hat{k}$

$=16(-\hat{i}+3\hat{j}+2\hat{k})$

Point of Intersection of lines :

$x=2\lambda+1,\ y=-\lambda,\ z=\lambda-3$

$x=\delta+4,\ y=\delta-3,\ z=2\delta-3$

So,

$2\lambda+1=\delta+4,\ -\lambda=\delta-3,\ \lambda-3=2\delta-3$

$\Rightarrow$ $2\lambda-\delta=3,\ \lambda+\delta=3,\ \lambda=2\delta$

So,

$\delta=1,\ \lambda=2$

So,

$P\equiv(5,-2,-1)$

So,

$-1(x-5)+3(y+2)+2(z+1)=0$ is plane

$\Rightarrow$ $-x+3y+2z+13=0 \Rightarrow x-3y-2z-13=0$

16. (a)

$\{x : [\sin^{-1} x] > [\cos^{-1} x]\} = [\sin 1, 1]$

Range of $[|\sin x| + |\cos x|] = \{0, 1\}$

Range of $[x + 1/2] + [x - 1/2] + 2[-x] = \{-3, -1\}$

Range of $\left[\frac{1}{[x-3]}\right] = \{-1, 0, 1\}$.

17. (d)

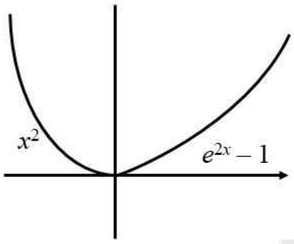
$$f_4(x) = \begin{cases} f_2(f_1(x)) \; ; \; x < 0 \\ f_2(f_1(x)) - 1 \; ; \; x \geq 0 \end{cases}$$

$$= \begin{cases} f_2(|x|) \; ; \; x < 0 \\ f_2(e^x) - 1 \; ; \; x \geq 0 \end{cases}$$

$$= \begin{cases} |x|^2 \; ; \; x < 0 \\ e^{2x} - 1 \; ; \; x \geq 0 \end{cases}$$

$$= \begin{cases} x^2 \; ; \; x < 0 \\ e^{2x} - 1 \; ; \; x \geq 0 \end{cases}$$

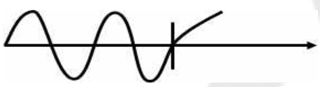
$f_4(x)$ is onto but not one-one



$f_3(x)$ is differentiable but not one-one

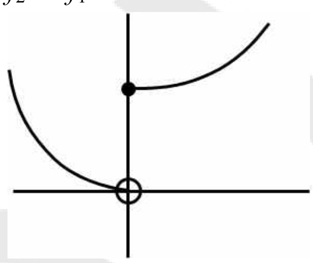
$f_3'(0+) = 1$

$f_3'(0-) = \cos 0 = 1$

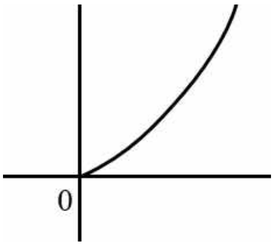


$$f_2 \, o \, f_1(x) = f_2 \begin{cases} |x|, \; x < 0 \\ e^x, \; x \geq 0 \end{cases} = \begin{cases} x^2, \; x < 0 \\ e^{2x}, \; x \geq 0 \end{cases}$$

$f_2 \; o \; f_1$ neither continuous nor one-one



$f_2(x) = x^2$ It is continuous and one-one



18. (d)

(P) $\int_0^1 f(x)dx = 1$

$f(x) = ax^2 + bx$ ($\because$, $f(0) = 0 \Rightarrow c = 0$)

$\Rightarrow \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^1 = 1$

$\Rightarrow \frac{a}{3} + \frac{b}{2} = 1$

$\Rightarrow 2a + 3b = 6$

If $a = 0$, $b = 2$; $a = 3$, $b = 0$

$(P) \rightarrow (2)$

(Q) $f(x) = \sin(x^2) + \cos(x^2)$

$= \sqrt{2}\sin(x^2 + \pi/4)$

for $f(x)$ = maximum, $f(x) = \sqrt{2}$

So, $\sin(x^2 + \pi/4) = 1 = \sin \pi/2$

$\Rightarrow x^2 + \pi/4 = n\pi + (-1)^n \pi/2$

$\Rightarrow x^2 = n\pi + (-1)^n \pi/2 - \pi/4$

$n = 0 : x^2 = \pi/4 \Rightarrow x = \pm\frac{\sqrt{\pi}}{2}$

$n = 1 : x^2 = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$

$n = 2 : x^2 = 2\pi + \frac{\pi}{4} = \frac{9\pi}{4} \Rightarrow x = \pm\frac{3}{2}\sqrt{\pi}$

$n = 3 : x^2 = 3\pi - \frac{3\pi}{4} = \frac{9\pi}{4}$

$n = 4 : x^2 = 4\pi + \frac{\pi}{4} = \frac{17\pi}{4}$ (not in range)

$n = -1; x^2 = -\pi - \frac{3\pi}{4}$ (can't accept)

So (Q) $\rightarrow$ (3)

(R)I $= \int_{-2}^{2}\frac{3x^2}{(1+e^x)}dx = \int_{-2}^{2}\frac{3x^2}{1+e^{-x}}dx$ $\left(\because, \int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx\right)$

$= \int_{-2}^{2}\frac{(e^x)3x^2}{1+e^x}dx$

So, $2I = \int_{-2}^{2}\frac{3x^2(1+e^x)}{(1+e^x)}dx - [x^3]_{-2}^{2} = 16$

$\Rightarrow$ I = 8

(R) $\rightarrow$ (1)

(S) $\dfrac{\displaystyle\int_{-1/2}^{1/2} \cos 2x \, \log\left(\frac{1+x}{1-x}\right)dx}{\displaystyle\int_{0}^{1/2} \cos 2x \, \log\left(\frac{1+x}{1-x}\right)dx}$

Numerator $= \displaystyle\int_{-1/2}^{1/2} \cos 2x \, \log\left(\frac{1-x}{1+x}\right) dx$ (By $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx$)

So, $I = \displaystyle\int_{-1/2}^{1/2} \cos 2x \, \log\left(\frac{1+x}{1-x}\right)dx$

$= \displaystyle\int_{-1/2}^{1/2} \cos 2x \, \log\left(\frac{1-x}{1+x}\right)dx$

$= -\displaystyle\int_{-1/2}^{1/2} \cos 2x \, \log\left(\frac{1+x}{1-x}\right)dx$

$= -I$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

So, given function value = 0 (S) $\to$ 4.